A new approach to the nonlinear connection associated with second-order (and higher-order) differential equation fields

1997 J. Phys. A: Math. Gen. 30 1739

(http://iopscience.iop.org/0305-4470/30/5/034)

J. Phys. A: Math. Gen. **30** (1997) 1739–1743. Printed in the UK PII: S0305-4470(97)77776-X

A new approach to the nonlinear connection associated with second-order (and higher-order) differential equation fields

D J Saunders
BBC Production Centre, The Open University, Walton Hall, Milton Keynes MK7 6BH, UK

Received 11 September 1996

Abstract. Each second-order (or higher-order) differential equation field defined on a tangent bundle or a jet bundle has an associated (possibly nonlinear) connection. We give a new geometric construction for this connection which uses a single straightforward formula in all cases; the fractional coefficients which complicate the traditional tensor formulæ are not required.

1. Introduction

Every second-order differential equation field (SODE field) Γ is associated with a nonlinear connection on the tangent (or jet) bundle of the associated configuration space. This has been known for a long time in the particular case where the SODE field is the geodesic spray of a Riemannian metric: in this case, the connection is just the metric connection. More recently, it has been shown that such an association always holds [1]. The properties of this connection have been studied by many authors, and have been used to investigate, for example, the separability of the original equations in appropriate coordinate systems [6], and the existence of a Lagrangian function for which the equations are the Euler–Lagrange equations [3].

In the autonomous case, the connection is defined on the tangent bundle $TQ \to Q$ and is usually specified by giving its horizontal projector as

$$P_{\mathrm{H}} = \tfrac{1}{2}(I - \mathcal{L}_\Gamma S)$$

where S is the type $(1, 1)$ tensor field representing the canonical almost tangential structure (or vertical endomorphism) on TQ. If local coordinates on Q are q^i and those on TQ are $(q^i, \dot{q}^i)$ then

$$S = \mathrm{d}q^i \otimes \frac{\partial}{\partial \dot{q}^i}$$

so if the SODE field is given by

$$\Gamma = \dot{q}^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial \dot{q}^i}$$

then

$$P_{\mathrm{H}} = \mathrm{d}q^i \otimes \left(\frac{\partial}{\partial q^i} + \frac{1}{2} \frac{\partial f^j}{\partial \dot{q}^i} \frac{\partial}{\partial \dot{q}^j} \right).$$

In the time-dependent case, the connection is defined instead on a jet bundle $J^1\pi \to E$, where π is a fibration $\pi{:}E \to \mathbb{R}$: if π has a given trivialization then we may think of E as a product manifold $\mathbb{R} \times Q$. In this case the horizontal projector is given by

$$P_{\mathrm{H}} = \tfrac{1}{2}(I - \mathcal{L}_\Gamma S + \mathrm{d}t \otimes \Gamma).$$

If local coordinates on E are (t, q^i) and those on $J^1\pi$ are $(t, q^i, \dot{q}^i)$ then

$$S = (\mathrm{d}q^i - \dot{q}^i\, \mathrm{d}t) \otimes \frac{\partial}{\partial \dot{q}^i}$$

so if the SODE field is given by

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial \dot{q}^i}$$

then

$$P_{\mathrm{H}} = \mathrm{d}q^i \otimes \left(\frac{\partial}{\partial q^i} + \frac{1}{2}\frac{\partial f^j}{\partial \dot{q}^i}\frac{\partial}{\partial \dot{q}^j} \right) + \mathrm{d}t \otimes \left(\frac{\partial}{\partial t} + \left(f^j - \frac{1}{2}\dot{q}^i \frac{\partial f^j}{\partial \dot{q}^i} \right) \frac{\partial}{\partial \dot{q}^j} \right).$$

Similar definitions may be given for higher-order differential equation fields, although the formulæ are more complicated [2]. For instance, if Γ is a $(k+1)$th-order differential equation field defined on T^kQ then the associated connection is defined on the bundle $T^kQ \to T^{k-1}Q$ and its horizontal projector is given by

$$P_{\mathrm{H}} = \frac{1}{k+1}(kI - \mathcal{L}_\Gamma S)$$

where now S is the higher-order type $(1, 1)$ tensor field defined canonically on the manifold T^kQ.

The purpose of the present note is to describe a new geometric construction for these connections. The construction differs from the existing ones in that it uses the S tensor defined on $T^{k+1}Q$ (or $J^{k+1}\pi$) rather than on T^kQ (or $J^k\pi$) for a $(k+1)$th-order differential equation field. It is a unifying and conceptually straightforward approach, where a single formula (without fractional coefficients) applies in all cases. We describe the construction in section 3; this description is preceded, in section 2, by an analogy which demonstrates the origin of the idea.

2. An analogy: systems with non-holonomic constraints

Suppose given a manifold E with fibration $\pi{:}E \to \mathbb{R}$. A certain type of non-holonomic constraint may be described by a submanifold $C \subset J^1\pi$ which projects onto E; several authors [4, 5, 8] have considered the sub-bundle

$$i^*S^*T^\circ C \subset T^*C$$

(or its equivalent in the autonomous case). To explain this construction,

— $i{:}C \to J^1\pi$ is the inclusion;

— $T^\circ C$ is the annihilator of TC in $i^*TJ^1\pi$, where $i^*TJ^1\pi$ is the pull-back bundle of $TJ^1\pi$ over C; and

— S^* represents the pointwise action of the S tensor (on $J^1\pi$) on cotangent vectors.

In [5] this sub-bundle is called the Chetaev bundle.

Locally, we may always choose coordinates (t, q^α, q^a) on E so that the constraints may be expressed in solved form as $\dot{q}^a = g^a(t, q^\beta, q^b, \dot{q}^\beta)$. The sub-bundle described above is then generated locally by the 1-forms

$$\eta^a = \mathrm{d}q^a - \frac{\partial g^a}{\partial \dot{q}^\alpha}\,\mathrm{d}q^\alpha - \left(g^a - \dot{q}^\alpha \frac{\partial g^a}{\partial \dot{q}^\alpha} \right) \mathrm{d}t.$$

If the constraints are affine, say $g^a(t, q^\beta, q^b, \dot{q}^\beta) = \dot{q}^\alpha B^a_\alpha(t, q^\beta, q^b) + B^a(t, q^\beta, q^b)$, then the 1-forms η^a will be given by

$$\eta^a = \mathrm{d}q^a - B^a_\alpha(t, q^\beta, q^b)\,\mathrm{d}q^\alpha - B^a(t, q^\beta, q^b)\,\mathrm{d}t$$

and will be basic over E, so that they will be pulled back from forms η^a_0 on E with the same coordinate representation. If the constraints are not affine then the forms η^a will be semi-basic over E.

Now suppose there is another fibration of E given by $\rho{:}E \to M$ where $\pi_0{:}M \to \mathbb{R}$ satisfies $\pi_0 \circ \rho = \pi$. Suppose that, with coordinates (t, q^α) on M and (t, q^α, q^a) on E, the constraints take the solved form above. In the affine case, the forms η^a_0 will be complementary to the horizontal forms $\mathrm{d}t$, $\mathrm{d}q^\alpha$ of the new projection ρ, and so they will be the vertical forms of a connection on ρ which we may represent by a map $\sigma{:}E \to J^1\rho$: we have $\sigma^a_\alpha = B^a_\alpha$ and $\sigma^a = B^a$. (Details of a similar construction may be found in [7].) In the general case, whether the constraints are affine or not, the forms η^a will be complementary to the horizontal forms $\mathrm{d}t$, $\mathrm{d}q^\alpha$, $\mathrm{d}\dot{q}^\alpha$ of the bundle $j^1\rho|_C{:}C \to J^1\pi_0$, and so they will be the vertical forms of a connection on $j^1\rho|_C$ with

$$\sigma^a_\alpha = \frac{\partial g^a}{\partial \dot{q}^\alpha} \qquad \sigma^a = g^a - \dot{q}^\alpha \frac{\partial g^a}{\partial \dot{q}^\alpha} \qquad \dot{\sigma}^a_\alpha = 0.$$

In either case, the observation is that, by starting with a submanifold of a jet (or tangent) manifold and a suitable fibration, we can generate the vertical forms of a connection by applying the S tensor to the annihilator of the submanifold's tangent bundle and then pulling the result back to the submanifold. We shall use this same technique to construct the connection associated with an arbitrary differential equation field.

3. The construction

A time-dependent $(k+1)$th-order vector field Γ is a section of $TJ^k\pi \to J^k\pi$ with the properties that it projects to $\partial/\partial t$ on $\mathbb{R}$ (and hence takes its values in $J^1\pi_k \subset TJ^k\pi$), and that it annihilates contact forms (so that it actually takes its values in $J^{k+1}\pi \subset J^1\pi_k$). Let γ be the section of $J^{k+1}\pi \to J^k\pi$ determined by the vector field Γ, and consider

$$V_\gamma := \gamma^* S^*_{k+1} T^\circ(\mathrm{Im}(\gamma)).$$

Here:

— $\mathrm{Im}(\gamma)$ is a submanifold of $J^{k+1}\pi$, and $i{:}\mathrm{Im}(\gamma) \to J^{k+1}\pi$ is the inclusion;

— $T(\mathrm{Im}(\gamma))$ is a sub-bundle of the pull-back bundle $i^*TJ^{k+1}\pi$ over $\mathrm{Im}(\gamma)$;

— $T^\circ(\mathrm{Im}(\gamma))$ is the annihilator of $T(\mathrm{Im}(\gamma))$ and is a sub-bundle of the pull-back bundle $i^*T^*J^{k+1}\pi$ over $\mathrm{Im}(\gamma)$;

— S_{k+1} is the S-tensor on $J^{k+1}\pi$, with S^*_{k+1} being the corresponding pointwise operator on cotangent vectors, so that $S^*_{k+1}T^\circ(\mathrm{Im}(\gamma))$ is another sub-bundle of $i^*T^*J^{k+1}\pi$ over $\mathrm{Im}(\gamma)$; and finally

— $V_\gamma = \gamma^* S^*_{k+1} T^\circ(\mathrm{Im}(\gamma))$ is a sub-bundle of $T^*J^k\pi$ which is just the vertical cotangent bundle of the nonlinear connection on $\pi_{k,k-1}{:}J^k\pi \to J^{k-1}\pi$.

Of course we need to prove this last assertion. We may check, using the properties of the appropriate S and the fact that γ is a section, that $\dim V_\gamma = \dim E - 1$ at each point, and that $V_\gamma \cap \pi^*_{k,k-1}T^*J^{k-1}\pi$ is just the zero section, so that V_γ is complementary to the horizontal cotangent bundle of $\pi_{k,k-1}$ and so does indeed determine a connection. To see that it is the same as the one given in earlier works, we just need to check the coordinate representation. As an illustration, we shall do this explicitly for the second-order case.

As before, take

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + f^i \frac{\partial}{\partial \dot{q}^i}$$

so that $\mathrm{Im}(\gamma)$ is the submanifold defined by $\ddot{q}^i = f^i$. Then $T^\circ(\mathrm{Im}(\gamma))$ is generated locally by the 1-forms

$$\mathrm{d}\ddot{q}^i - \frac{\partial f^i}{\partial \dot{q}^j} \mathrm{d}\dot{q}^j - \frac{\partial f^i}{\partial q^j} \mathrm{d}q^j - \frac{\partial f^i}{\partial t} \mathrm{d}t.$$

The S tensor on $J^2\pi$ is given by

$$S_2 = (\mathrm{d}q^i - \dot{q}^i \,\mathrm{d}t) \otimes \frac{\partial}{\partial \dot{q}^i} + 2(\mathrm{d}\dot{q}^i - \ddot{q}^i \,\mathrm{d}t) \otimes \frac{\partial}{\partial \ddot{q}^i}$$

so that $S_2^* T^\circ(\mathrm{Im}(\gamma))$ is locally generated by

$$\begin{aligned} 2(\mathrm{d}\dot{q}^i - \ddot{q}^i \,\mathrm{d}t) &- \frac{\partial f^i}{\partial \dot{q}^j}(\mathrm{d}q^j - \dot{q}^j \,\mathrm{d}t) \\ &= 2\left(\mathrm{d}\dot{q}^i - \frac{1}{2}\frac{\partial f^i}{\partial \dot{q}^j}\mathrm{d}q^j - \left(\ddot{q}^i - \frac{1}{2}\dot{q}^j \frac{\partial f^i}{\partial \dot{q}^j}\right)\mathrm{d}t\right). \end{aligned}$$

Finally, therefore, $\gamma^* S_2^* T^\circ(\mathrm{Im}(\gamma))$ is locally generated by (omitting the initial factor of 2)

$$\mathrm{d}\dot{q}^i - \frac{1}{2}\frac{\partial f^i}{\partial \dot{q}^j}\mathrm{d}q^j - \left(f^i - \frac{1}{2}\dot{q}^j \frac{\partial f^i}{\partial \dot{q}^j}\right)\mathrm{d}t$$

which may be recognized as the vertical forms of the SODE connection. Higher-order cases may be checked in exactly the same way.

An analogous construction may be performed in the autonomous case, where a $(k+1)$th-order vector field Γ gives rise to a section γ of $T^{k+1}Q \to T^kQ$: here, $V_\gamma = \gamma^* S_{k+1}^* T^\circ(\mathrm{Im}(\gamma))$ will be a sub-bundle of T^*T^kQ and will be the vertical cotangent bundle of the nonlinear connection on $T^kQ \to T^{k-1}Q$.

One last remark is that, if we try this technique for a first-order vector field, we find that the vertical forms are just $\mathrm{d}q^i - f^i \,\mathrm{d}t$ in the time-dependent case, so that the (single) horizontal vector field of the connection is Γ itself; in the autonomous case all the forms $\mathrm{d}q^i$ are vertical so the connection is vacuous.

Acknowledgments

This idea originated during the course of work on non-holonomic systems with Willy Sarlet and Frans Cantrijn, and subsequent discussions with Alberto Ibort. I should like to thank them for their encouragement and their hospitality. I also wish to acknowledge support from Universiteit Gent and Universidad Complutense de Madrid.

References

[1] Crampin M 1983 Tangent bundle geometry for Lagrangian dynamics *J. Phys. A: Math. Gen.* **16** 3755–72

[2] Crampin M, Sarlet W and Cantrijn F 1986 Higher-order differential equations and higher-order Lagrangian mechanics *Math. Proc. Camb. Phil. Soc.* **99** 565–87

[3] Crampin M, Sarlet W, Martínez E, Byrnes G B and Prince G 1994 Towards a geometrical understanding of Douglas's solution of the inverse problem of the calculus of variations *Inverse Problems* **10** 245–60

[4] Giachetta G 1992 Jet methods in nonholonomic mechanics *J. Math. Phys.* **33** 1652–65

[5] Massa E and Pagani E 1995 A new look at classical mechanics of constrained systems *Preprint*

[6] Martínez E, Cariñena J F and Sarlet W 1993 Geometric characterization of separable second-order differential equations *Math. Proc. Camb. Phil. Soc.* **113** 205–24

[7] Sarlet W, Cantrijn F and Saunders D J 1995 A geometrical framework for the study of non-holonomic Lagrangian systems *J. Phys. A: Math. Gen.* **28** 3253–68

[8] Vershik A M 1984 Classical and non-classical dynamics with constraints *Global Analysis—Studies and Applications I (Lecture Notes in Math. 1108)* ed Yu G Borisovich and Yu E Gliklikh (Berlin: Springer) pp 278–301